Int. J. Solids Structures, 1967, Vol. 3, pp. 1 to 22. Pergamon Press Ltd. Printed in Great Britain

ON THE GRIFFITH ENERGY CRITERION FOR BRITTLE FRACTURE*

G. C. SIH

Lehigh University, Bethlehem, Pa.

and

H. LIEBOWITZ

Structural Mechanics Branch, Office of Naval Research; and
The Catholic University of America, Washington, D.C.

Abstract—The Griffith theory for unstable crack propagation is re-examined to investigate the difference of opinion as to the precise dependency of critical stress upon the elastic constant in a brittle material. The apparent disagreement arising from the calculation of strain energy stored in a cracked body loaded at infinity is resolved by the observation that the energy of the crack or cavity of certain shape is reasonably geometry independent. It is found from the solution of the concentric-annulus problem that the stresses and displacements on a closed contour about the cavity must be modified to yield the correct form of elastic energy. Clapeyron's theorem is employed so that the energy function may be derived from the work done by surface tractions. A general method for determining the strain energy in an infinite medium with cavities of arbitrary configuration is presented. Closed form solutions to the problem of an elliptically-shaped flaw are obtained and incorporated into a theory of brittle fracture. More specifically, critical stresses for an elliptical flaw or crack owing to biaxial tension and pure shear are provided. The present analysis also confirms Griffith's claim in 1924 that his original energy expression, published in 1921, is indeed erroneous.

INTRODUCTION

FOR more than four decades ago, Griffith [1] suggested an energy approach to predict the fracture strength of elastic solids containing crack-like flaws. Using Inglis's solution [2] for the problem of an elliptical hole in a plate subjected to all around tension, he computed the elastic energy due to the presence of a crack, and compared the amount of increase of this energy as the crack grows with the energy required to form the new fracture surfaces. He then found that the critical stress needed for the failure of a crack of length $2a$ is

$$\sigma_{cr} = \sqrt{\left[\frac{8\gamma E}{\pi a(1+\nu)(3-\kappa)}\right]}, \qquad (1921) \tag{1}$$

where γ is the specific surface energy, E is Young's modulus, and ν is Poisson's ratio. The elastic constant κ takes the value of $3-4\nu$ for plane strain and $(3-\nu)/(1+\nu)$ for generalized plane stress. In 1924, Griffith [3] claimed the calculation of strain energy leading to equation (1) was erroneous in that, quoting from his classical paper,

* This work was sponsored by the U.S. Navy through the Office of Naval Research under Contract Nonr-610(06).

"the expressions used for the stresses gave values at infinity differing from the postulated uniform stress at infinity by an amount which, though infinitesimal, yet made a finite contribution to the energy when integrated round the infinite boundary. This difficulty has been overcome by slightly modifying the expressions for the stresses, so as to make this contribution to the energy vanish."

He revised equation (1) to read as

$$\sigma_{cr} = \sqrt{\left[\frac{8\gamma E}{\pi a(1+\nu)(\kappa+1)}\right]}, \qquad (1924) \tag{2}$$

but the details of his calculations were not published. As a consequence, many of the investigators in fracture mechanics have attempted to fill in the missing details and have even cast doubt on such a revision.

While the critical stresses in equations (1) and (2) differ only in the quantity containing κ, the physical interpretation of the fracture criterion can change significantly for different loading conditions. In the case of biaxial tension at infinity, the 1921 version would predict that the applied stress parallel to the crack may either strengthen or weaken the body. The fracture strength of an infinite medium subjected to the conditions of plane strain may be infinite if the material is incompressible and is in the state of pure shear at large distance away from the crack. These and similar view points have been supported by Wolf [4], Smekal [5], Berry [6], Swedlow [7], and others. In general, however, Griffith's modified work in 1924 has been accepted in the literature as being correct. In view of this apparent conflict, the theory of brittle fracture has yet to rest on a firm foundation.

Recently, Spencer [8] has made an effort to explain Griffith's work. He argued that if the proper tractions are specified at the boundary of a large circle around the crack to within quantities of a certain order, then it is possible to verify Griffith's 1924 results. It must be emphasized here that the crucial question to be answered is not how Griffith arrived at his end results but rather to prove whether his energy expressions are determined correctly or not?

At first, it seems that the argument could be settled by computing for the work done by tractions applied to the outer boundary of the region bounded by confocal ellipses. The outer ellipse is then permitted to grow without limit, while the inner ellipse degenerates into a sharp crack. However, the mathematics involved for a closed form solution of the problem of confocal ellipses do not appear tractable. An alternative approach is to establish the qualitative behavior of the strain energy function by solving a relatively simple problem of a concentric annulus. In fact, the resemblance of the expressions for the excess of strain energy due to a crack and a circular hole is remarkable. If uniform stress, σ, is applied in all directions at infinity, the energy of the crack having length $2a$ is

$$\Delta W = \frac{\pi(1+\nu)\sigma^2 a^2}{4E} \cdot \begin{cases} (3-\kappa), & (1921) \quad \text{(3a)} \\ (\kappa+1), & (1924) \quad \text{(3b)} \end{cases}$$

and of the circular hole with diameter $2a$ is

$$\Delta W = \frac{\pi(1+\nu)\sigma^2 a^2}{2E} \cdot \begin{cases} (3-\kappa), & (1921) \quad \text{(4a)} \\ (\kappa+1), & (1924) \quad \text{(4b)} \end{cases}$$

The only difference is a factor of $\frac{1}{2}$. Hence, the concentric-annulus problem may be used as the model to establish procedures for finding the strain energy in an infinite body.

It is intended to develop a method for calculating the work done or strain energy in an infinite body with a cavity of any shape when external loads are specified at infinity. This is accomplished by modifying the stresses and displacements on a large contour around the cavity in accordance with the results obtained from the concentric-annulus problem. The necessary modifications are then applied to the problem of an infinite medium weakened by an elliptical hole. The variation of critical stress with the change in geometry of the ellipse is also studied.

CIRCULAR FLAW IN INFINITE MEDIUM

For clarity, the circular flaw in an infinite body will be adopted as the model to demonstrate the way in which strain energy was calculated by Griffith in 1921. The edge of the flaw is free from normal and shear stresses. At infinity, the conditions

$$\sigma_x = e\sigma, \qquad \sigma_y = \sigma, \qquad \tau_{xy} = \tau, \quad \text{as} \quad (x^2+y^2)^{1/2} \to \infty \tag{5}$$

are to be satisfied. The amount of tension or compression in the horizontal direction is controlled through the parameter e. By application of Clapeyron's theorem [9], the total strain energy in the two-dimensional elastic body as shown in Fig. 1 may be represented by the integral

$$W = \frac{1}{2}\int_0^{2\pi} [\sigma_r u_r + \tau_{r\theta} v_\theta]_{r=c}\, c\, \mathrm{d}\theta \tag{6}$$

where c is the radius of a large circular contour surrounding the hole of radius a. The

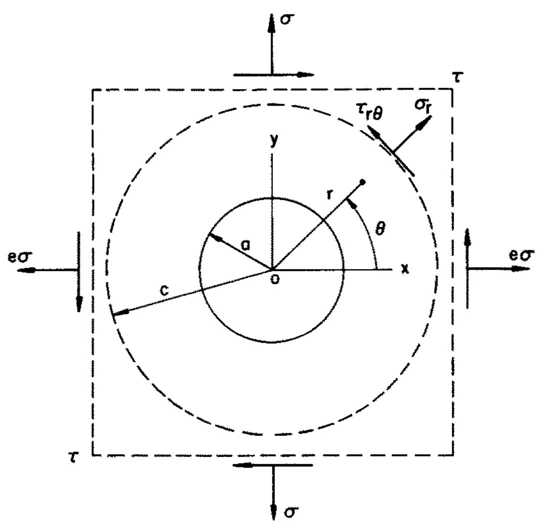

FIG. 1. Circular flaw in infinite medium.

stresses $\sigma_r, \tau_{r\theta}$, and displacements u_r, v_θ are referred to polar coordinates r, θ. The boundary conditions in equation (5) may be considered as the sum of two stress systems, namely biaxial tension and pure shear, which will be treated separately.

The solution to the problem of a circular hole in an infinite body under biaxial tension is well known and can be found in any text book on elasticity. It is also given in terms of complex potentials by equations (76) in Appendix 2. To provide continuity with the development to follow, the stresses and displacements will be presented. They are

$$\sigma_r = \frac{\sigma}{2}\left[1-\left(\frac{a}{r}\right)^2\right]\left\{(1+e)-\left[1-3\left(\frac{a}{r}\right)^2\right](1-e)\cos 2\theta\right\}, \tag{7a}$$

$$\tau_{r\theta} = \frac{\sigma}{2}\left[1-\left(\frac{a}{r}\right)^2\right]\left[1+3\left(\frac{a}{r}\right)^2\right](1-e)\sin 2\theta, \tag{7b}$$

and

$$u_r = \frac{\sigma}{4\mu}r\left\{\frac{1}{2}\left[(\kappa-1)+2\left(\frac{a}{r}\right)^2\right](1+e)-\left[1+(\kappa+1)\left(\frac{a}{r}\right)^2-\left(\frac{a}{r}\right)^4\right](1-e)\cos 2\theta\right\}. \tag{8a}$$

$$v_\theta = \frac{\sigma}{4\mu}r\left[1+(\kappa-1)\left(\frac{a}{r}\right)^2+\left(\frac{a}{r}\right)^4\right](1-e)\sin 2\theta, \tag{8b}$$

in which μ is the shear modulus. The unboundness of the displacements at $r = \infty$ is a peculiarity of elasticity problems with prescribed loads at infinity and cannot be removed. Inserting equations (7) and (8) into (6) and letting $r = c$, the stored energy is

$$W = \frac{\pi(1+\nu)\sigma^2c^2}{8E}\left[1-\left(\frac{a}{c}\right)^2\right]\left\{\left[(\kappa-1)+2\left(\frac{a}{c}\right)^2\right](1+e)^2 + 2\left[1+\kappa\left(\frac{a}{c}\right)^2-3\left(\frac{a}{c}\right)^4+3\left(\frac{a}{c}\right)^6\right](1-e)^2\right\}. \tag{9}$$

When c becomes infinitely large, equation (9) tends toward the value

$$W = \frac{\pi(1+\nu)\sigma^2c^2}{8E}\left\{(\kappa-1)(1+e)^2+2(1-e)^2+|(3-\kappa)(1+e)^2+2(\kappa-1)(1-e)^2| \times\left(\frac{a}{c}\right)^2+O\left[\left(\frac{a}{c}\right)^4\right]\right\}.$$

The term

$$W_0 = \frac{\pi(1+\nu)\sigma^2c^2}{8E}[(\kappa-1)(1+e)^2+2(1-e)^2], \qquad \text{(tension)} \tag{10}$$

is expected to increase without bound as $c \to \infty$ since it represents the energy in an infinite body with no hole present. The contribution due to the presence of the circular hole is

$$\Delta W = W - W_0 = \frac{\pi(1+\nu)\sigma^2a^2}{8E}[(3-\kappa)(1+e)^2+2(\kappa-1)(1-e)^2], \qquad (1921) \tag{11}$$

and remains finite. For a uniformly stressed body, $e = 1$, equation (11) reduces to equation (4a). A similar expression for the crack problem is

$$\Delta W = \frac{\pi(1+\nu)\sigma^2a^2}{8E}[(3-\kappa)(1+e)+(\kappa-1)(1-e)], \qquad (1921) \tag{12}$$

which agrees with equation (3a) for $e = 1$. Equation (12) will be deduced subsequently from the solution of the elliptical-cavity problem. Note that ΔW depends sensitively upon the applied stress parallel to the line crack and it changes sign for sufficiently large value of e.

From equations (78) in Appendix 3, the stress and displacement fields for the case of pure shear can be derived:

$$\sigma_r = \tau\left[1-\left(\frac{a}{r}\right)^2\right]\left[1-3\left(\frac{a}{r}\right)^2\right]\sin 2\theta, \tag{13a}$$

$$\tau_{r\theta} = \tau\left[1-\left(\frac{a}{r}\right)^2\right]\left[1+3\left(\frac{a}{r}\right)^2\right]\cos 2\theta, \tag{13b}$$

and

$$u_r = \frac{\tau}{2\mu}r\left[1+(\kappa+1)\left(\frac{a}{r}\right)^2-\left(\frac{a}{r}\right)^4\right]\sin 2\theta, \tag{14a}$$

$$v_\theta = \frac{\tau}{2\mu}r\left[1+(\kappa-1)\left(\frac{a}{r}\right)^2+\left(\frac{a}{r}\right)^4\right]\cos 2\theta. \tag{14b}$$

It follows that the total energy is

$$W = \frac{\pi(1+\nu)\tau^2c^2}{E}\left[1-\left(\frac{a}{c}\right)^2\right]\left[1+\kappa\left(\frac{a}{c}\right)^2-3\left(\frac{a}{c}\right)^4+3\left(\frac{a}{c}\right)^6\right]. \tag{15}$$

Expanding equation (15) for $c \gg a$ yields

$$W = \frac{\pi(1+\nu)\tau^2c^2}{E}\left\{1+(\kappa-1)\left(\frac{a}{c}\right)^2+O\left[\left(\frac{a}{c}\right)^4\right]\right\}.$$

The first term

$$W_0 = \frac{\pi(1+\nu)\tau^2c^2}{E}, \qquad \text{(shear)} \tag{16}$$

represents the energy in the body with no cavity and the second term

$$\Delta W = \frac{\pi(1+\nu)\tau^2a^2}{E}(\kappa-1), \qquad (1921) \tag{17}$$

is that part of the energy due to the presence of the circular cavity. If the same calculations were carried out for a cavity in the form of a crack, it can be shown that

$$\Delta W = \frac{\pi(1+\nu)\tau^2a^2}{4E}(\kappa-1), \qquad (1921). \tag{18}$$

As pointed out in [7], the shear energy of a crack vanishes for $\kappa \doteq 3$–4ν and $\nu = \frac{1}{2}$.

The validity of equations (11) and (17) will now be checked by solving the problem of concentric annulus.

CONCENTRIC ANNULUS

Let an elastic body be bounded by two concentric circles of outer radius c and inner radius a, Fig. 2. The origin of coordinates is placed at the center of the circles and the applied surface tractions are

$$\sigma_r = \frac{\sigma}{2}[(1+e)-(1-e)\cos 2\theta]+\tau \sin 2\theta, \qquad r = c \tag{19a}$$

$$\tau_{r\theta} = \frac{\sigma}{2}(1-e)\sin 2\theta+\tau \cos 2\theta, \qquad r = c. \tag{19b}$$

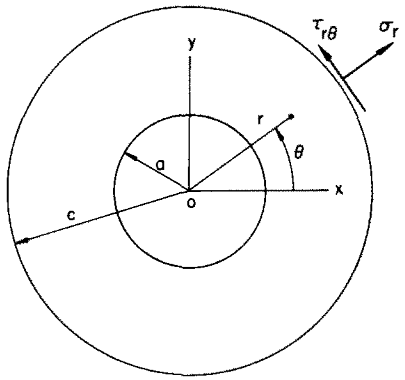

FIG. 2. Plane extension of circular ring.

Equations (19) are obtained from equations (5) by transforming the coordinate system x, y to r, θ. The tractions on the inner boundary, $r = a$, are assumed to vanish. Although this problem has not been solved previously, the solution can be found without difficulty. The details of the derivations are outlined in the Appendices.

Taking $\tau = 0$ in equations (19), the stresses and displacements in the annulus may be calculated from the complex functions, equations (75) in Appendix 2, by application of the Kolosov–Muskhelishvili [10] stress combinations. The results are

$$\begin{aligned}\sigma_r = \frac{\sigma}{2}\left[1-\left(\frac{a}{c}\right)^2\right]^{-3}\Bigg\langle&\left[1-\left(\frac{a}{c}\right)^2\right]^2(1+e)-\left[1+\left(\frac{a}{c}\right)^2+4\left(\frac{a}{c}\right)^4\right](1-e)\cos 2\theta\\&-\left\{\left[1-\left(\frac{a}{c}\right)^2\right]^2(1+e)-4\left[1+\left(\frac{a}{c}\right)^2+\left(\frac{a}{c}\right)^4\right](1-e)\cos 2\theta\right\}\left(\frac{a}{r}\right)^2-3\left\{\left[1+\left(\frac{a}{c}\right)^2\right]\right.\\&\left.\times(1-e)\cos 2\theta\right\}\left(\frac{a}{r}\right)^4\Bigg\rangle,\end{aligned} \tag{20a}$$

$$\begin{aligned}\tau_{r\theta} = \frac{\sigma}{2}\left[1-\left(\frac{a}{c}\right)^2\right]^{-3}\Bigg\{&-6\left(\frac{a}{c}\right)^2\left(\frac{r}{c}\right)^2+\left[1+\left(\frac{a}{c}\right)^2+4\left(\frac{a}{c}\right)^4\right]+2\left[1+\left(\frac{a}{c}\right)^2+\left(\frac{a}{c}\right)^4\right]\left(\frac{a}{r}\right)^2\\&-3\left[1+\left(\frac{a}{c}\right)^2\right]\left(\frac{a}{r}\right)^4\Bigg\}(1-e)\sin 2\theta,\end{aligned} \tag{20b}$$

and

$$u_r = \frac{\sigma}{4\mu} r \left[1 - \left(\frac{a}{c}\right)^2\right]^{-3} \left\langle (3-\kappa)\left(\frac{a}{c}\right)^2 \left(\frac{r}{c}\right)^2 + \frac{1}{2}\left[1 - \left(\frac{a}{c}\right)^2\right]^2 (\kappa - 1)(1+e) \right.$$
$$- \left[1 + \left(\frac{a}{c}\right)^2 + 4\left(\frac{a}{c}\right)^4\right](1-e)\cos 2\theta + \left\{\left[1 - \left(\frac{a}{c}\right)^2\right]^2 (1+e) - \left[1 + \left(\frac{a}{c}\right)^2 + \left(\frac{a}{c}\right)^4\right]\right. \qquad \text{(21a)}$$
$$\left. \left. \times (\kappa+1)(1-e)\cos 2\theta \right\}\left(\frac{a}{r}\right)^2 + \left\{\left[1 + \left(\frac{a}{c}\right)^2\right](1-e)\cos 2\theta\right\}\left(\frac{a}{r}\right)^4 \right\rangle,$$

$$v_\theta = \frac{\sigma}{4\mu} r \left[1 - \left(\frac{a}{c}\right)^2\right]^{-3} \left\{ -(\kappa+3)\left(\frac{a}{c}\right)^2 \left(\frac{r}{c}\right)^2 + \left[1 + \left(\frac{a}{c}\right)^2 + 4\left(\frac{a}{c}\right)^4\right] \right.$$
$$\left. + \left[1 + \left(\frac{a}{c}\right)^2 + \left(\frac{a}{c}\right)^4\right](\kappa - 1)\left(\frac{a}{r}\right)^2 + \left[1 + \left(\frac{a}{c}\right)^2\right]\left(\frac{a}{r}\right)^4 \right\}(1-e)\sin 2\theta. \qquad \text{(21b)}$$

Upon substitution of equations (20) and (21) into (6) gives

$$W = \frac{\pi(1+\nu)\sigma^2 c^2}{8E}\left[1 - \left(\frac{a}{c}\right)^2\right]^{-3}\left\{\left[1 - \left(\frac{a}{c}\right)^2\right]\left[(\kappa-1) + 2\left(\frac{a}{c}\right)^2\right](1+e)^2\right.$$
$$\left. + 2\left[1 + (\kappa-2)\left(\frac{a}{c}\right)^2 + (\kappa+4)\left(\frac{a}{c}\right)^4 + \kappa\left(\frac{a}{c}\right)^6\right](1-e)^2\right\}. \qquad \text{(22)}$$

This is an exact expression of the energy accumulated within the annulus, where c is arbitrary. In the limit as $c \to \infty$, equation (22) renders the form of W for an infinite body with a circular hole, i.e.

$$W = \frac{\pi(1+\nu)\sigma^2 c^2}{8E}\left\{(\kappa-1)(1+e)^2 + 2(1-e)^2 + [(1+e)^2 + 2(1-e)^2]\right.$$
$$\left. \times (\kappa+1)\left(\frac{a}{c}\right)^2 + O\left[\left(\frac{a}{c}\right)^4\right]\right\}, \qquad c \gg a. \qquad \text{(23)}$$

In the absence of the hole, $a = 0$, equation (10) is recovered. Moreover, the energy of the circular hole can be extracted from equation (23):

$$\Delta W = \frac{\pi(1+\nu)\sigma^2 a^2}{8E}(\kappa+1)[(1+e)^2 + 2(1-e)^2], \qquad (1924) \qquad \text{(24)}$$

which is in serious disagreement with equation (11). Equation (24) is identified with Griffith's 1924 work, because of its striking similarity to the energy of the crack given by equation (3b). The relationship between equations (11) and (24) will be discussed later on.

Turning now to the problem of specifying $\sigma = 0$ in equations (19), use is made of equations (77) in Appendix 3. The stress components σ_r, $\tau_{r\theta}$ can be found as

$$\sigma_r = \tau\left[1 - \left(\frac{a}{c}\right)^2\right]^{-3}\left\{\left[1 + \left(\frac{a}{c}\right)^2 + 4\left(\frac{a}{c}\right)^4\right] - 4\left[1 + \left(\frac{a}{c}\right)^2 + \left(\frac{a}{c}\right)^4\right]\left(\frac{a}{r}\right)^2\right.$$
$$\left. + 3\left[1 + \left(\frac{a}{c}\right)^2\right]\left(\frac{a}{r}\right)^4\right\}\sin 2\theta, \qquad \text{(25a)}$$

$$\tau_{r\theta} = \tau\left[1-\left(\frac{a}{c}\right)^2\right]^{-3}\left\{-6\left(\frac{a}{c}\right)^2\left(\frac{r}{c}\right)^2+\left[1+\left(\frac{a}{c}\right)^2+4\left(\frac{a}{c}\right)^4\right]+2\left[1+\left(\frac{a}{c}\right)^2+\left(\frac{a}{c}\right)^4\right]\right.$$
$$\left.\times\left(\frac{a}{r}\right)^2-3\left[1+\left(\frac{a}{c}\right)^2\right]\left(\frac{a}{r}\right)^4\right\}\cos 2\theta, \tag{25b}$$

and the displacement components u_r, v_θ are given by

$$u_r = \frac{\tau}{2\mu}r\left[1-\left(\frac{a}{c}\right)^2\right]^{-3}\left\{(\kappa-3)\left(\frac{a}{c}\right)^2\left(\frac{r}{c}\right)^2+\left[1+\left(\frac{a}{c}\right)^2+4\left(\frac{a}{c}\right)^4\right]+\left[1+\left(\frac{a}{c}\right)^2+\left(\frac{a}{c}\right)^4\right]\right.$$
$$\left.\times(\kappa+1)\left(\frac{a}{r}\right)^2-\left[1+\left(\frac{a}{c}\right)^2\right]\left(\frac{a}{r}\right)^4\right\}\sin 2\theta, \tag{26a}$$

$$v_\theta = \frac{\tau}{2\mu}r\left[1-\left(\frac{a}{c}\right)^2\right]^{-3}\left\{-(\kappa+3)\left(\frac{a}{c}\right)^2\left(\frac{r}{c}\right)^2+\left[1+\left(\frac{a}{c}\right)^2+4\left(\frac{a}{c}\right)^4\right]+\left[1+\left(\frac{a}{c}\right)^2+\left(\frac{a}{c}\right)^4\right]\right.$$
$$\left.\times(\kappa-1)\left(\frac{a}{r}\right)^2+\left[1+\left(\frac{a}{c}\right)^2\right]\left(\frac{a}{r}\right)^4\right\}\cos 2\theta. \tag{26b}$$

On setting $r = c$ in equations (25), (26) and integrating, it is found from equation (6) that

$$W = \frac{\pi(1+\nu)\tau^2c^2}{E}\left[1-\left(\frac{a}{c}\right)^2\right]^{-3}\left[1+(\kappa-2)\left(\frac{a}{c}\right)^2+(4+\kappa)\left(\frac{a}{c}\right)^4+\kappa\left(\frac{a}{c}\right)^6\right]. \tag{27}$$

As the ratio $a/c \to 0$, equation (27) may be written in the form

$$W = \frac{\pi(1+\nu)\tau^2c^2}{E}\left\{1+(\kappa+1)\left(\frac{a}{c}\right)^2+O\left[\left(\frac{a}{c}\right)^4\right]\right\}.$$

The increase of strain energy owing to the cavity becomes

$$\Delta W = \frac{\pi(1+\nu)\tau^2a^2}{E}(\kappa+1), \qquad (1924). \tag{28}$$

Comparing ΔW in equation (28) with that shown in equation (17) for the same problem of pure shear, they obviously fail to agree. In the case of a crack, the expression for ΔW changes only slightly, i.e.

$$\Delta W = \frac{\pi(1+\nu)\tau^2a^2}{4E}(\kappa+1), \qquad (1924) \tag{29}$$

which differs from equation (18) in exactly the same way as equation (28) from (17).

In passing, it should be reminded that equations (24) and (28) represent the correct expressions of strain energy for the circular cavity problem of biaxial tension and pure shear applied at infinity, respectively, since they are obtained as limiting cases from the exact solutions of the concentric-annulus problem. Thus, there should be no doubt that equations (11) and (17) are erroneous. Furthermore, because the same discrepancies may be accounted for the crack problem, it is plausible to conjecture that the method of solution used for obtaining equations (12) and (18) is in error. The necessary modifications that must be added onto equations (12) and (18) will be derived in the next section.

PROPOSED METHOD OF SOLUTION

As mentioned earlier, it would be difficult, if not impossible, to check the correctness of equations (12) and (18) by solving the problem of confocal ellipses and then let the outer ellipse be infinitely large, mainly because the theory of plane elasticity prohibits closed form solutions to almost all problems with regions that are multiply-connected. An exception to this is the concentric-annulus problem whose solution plays an important role in the development of a method for finding the strain energy function.

Consider the discrepancy between equations (11) and (24) caused by the variance of stresses and displacements in equations (7), (8), (20), and (21). If the contour of radius c in Fig. 1 is made large as compared to the hole radius a, then equations (7) for $r = c$ become

$$\sigma_r^{(1)} = \frac{\sigma}{2}\left\{(1+e)-(1-e)\cos 2\theta-[(1+e)-4(1-e)\cos 2\theta]\left(\frac{a}{c}\right)^2+O\left[\left(\frac{a}{c}\right)^4\right]\right\}, \tag{30a}$$

$$\tau_{r\theta}^{(1)} = \frac{\sigma}{2}\left\{(1-e)\sin 2\theta+2[(1-e)\sin 2\theta]\left(\frac{a}{c}\right)^2+O\left[\left(\frac{a}{c}\right)^4\right]\right\}, \tag{30b}$$

and equations (8) take the forms

$$u_r^{(1)} = \frac{\sigma}{2\mu}c\left\{(\kappa-1)(1+e)-2(1-e)\cos 2\theta+[(1+e)-(\kappa+1)(1-e)\cos 2\theta]\left(\frac{a}{c}\right)^2+O\left[\left(\frac{a}{c}\right)^4\right]\right\}, \tag{31a}$$

$$v_\theta^{(1)} = \frac{\sigma}{2\mu}c\left\{2(1-e)\sin 2\theta+2[(\kappa-1)(1-e)\sin 2\theta]\left(\frac{a}{c}\right)^2+O\left[\left(\frac{a}{c}\right)^4\right]\right\}. \tag{31b}$$

In the same way for $c \gg a$ and $r = c$, the stresses in equations (20) simplify to

$$\sigma_r^{(2)} = \frac{\sigma}{2}[(1+e)-(1-e)\cos 2\theta], \tag{32a}$$

$$\tau_{r\theta}^{(2)} = \frac{\sigma}{2}(1-e)\sin 2\theta, \tag{32b}$$

and the displacements in equations (21) are

$$u_r^{(2)} = \frac{\sigma}{2\mu}c\left\{(\kappa-1)(1+e)-2(1-e)\cos 2\theta+[(\kappa+1)(1+e)-4(\kappa+1)(1-e)\cos 2\theta]\times\left(\frac{a}{c}\right)^2+O\left[\left(\frac{a}{c}\right)^4\right]\right\}, \tag{33a}$$

$$v_\theta^{(2)} = \frac{\sigma}{2\mu}c\left\{2(1-e)\sin 2\theta+O\left[\left(\frac{a}{c}\right)^4\right]\right\}. \tag{33b}$$

When a/c is identically zero, equations (30) and (31) agree with equations (32) and (33), respectively. In general, they must be distinguished by the superscripts (1) and (2) since the stored energy depends upon terms of order a^2/c^2 in the stresses and of order a^2/c in the displacements. The higher order terms in a/c do not contribute to the strain energy.

Comparing the two systems of stresses and displacements labeled by superscripts (1) and (2), and defining the difference between any two like quantities by the symbol Δ, such as $\Delta\sigma_r = \sigma_r^{(2)} - \sigma_r^{(1)}$, etc., it is found that

$$\Delta\sigma_r = \frac{\sigma}{2}\left\{[(1+e)-4(1-e)\cos 2\theta]\left(\frac{a}{c}\right)^2 + O\left[\left(\frac{a}{c}\right)^4\right]\right\}, \tag{34a}$$

$$\Delta\tau_{r\theta} = -\frac{\sigma}{2}\left\{[2(1-e)\sin 2\theta]\left(\frac{a}{c}\right)^2 + O\left[\left(\frac{a}{c}\right)^4\right]\right\}, \tag{34b}$$

and

$$\Delta u_r = \frac{\sigma}{2\mu}c\left\{[(\kappa-1)(1+e)-2(\kappa+1)(1-e)\cos 2\theta]\left(\frac{a}{c}\right)^2 + O\left[\left(\frac{a}{c}\right)^4\right]\right\}, \tag{35a}$$

$$\Delta v_\theta = -\frac{\sigma}{2\mu}c\left\{2[(\kappa-1)(1-e)\sin 2\theta]\left(\frac{a}{c}\right)^2 + O\left[\left(\frac{a}{c}\right)^4\right]\right\}. \tag{35b}$$

These stresses and displacements may be interpreted as corrections to equations (30) and (31). In addition, their character suggests the following generalization

$$\Delta\sigma_r = 2A - 2C\cos 2\theta, \tag{36a}$$

$$\Delta\tau_{r\theta} = (6Br^2 + 2C)\sin 2\theta, \tag{36b}$$

and

$$\Delta u_r = \frac{2}{\mu}r\{(\kappa-1)A + [(\kappa-3)Br^2 - 2C]\cos 2\theta\}, \tag{37a}$$

$$\Delta v_\theta = \frac{2}{\mu}r[(\kappa+3)Br^2 + 2C]\sin 2\theta, \tag{37b}$$

which are derived from the Goursat functions*

$$\phi(z) = (A + Bz^2)z, \qquad \psi(z) = 2Cz. \tag{38}$$

The complex variable z stands for $x+iy$. Equations (36) and (37) are valid for cavities of arbitrary shape as long as the radial distance r is large in comparison with the geometric dimensions of the cavity. The constants A, B, and C may be evaluated from the prescribed surface tractions. For instance, adding equations (36) onto equations (30) and applying the boundary conditions, equations (19), yield

$$A = \frac{\sigma}{4}(1+e)\left(\frac{a}{c}\right)^2, \qquad B = -\frac{\sigma}{2c^2}(1-e)\left(\frac{a}{c}\right)^2, \qquad C = \sigma(1-e)\left(\frac{a}{c}\right)^2. \tag{39}$$

Inserting these constants into equations (36) and (37), it can be shown that the resulting expressions are precisely those given by equations (34) and (35).

Without going into details, similar modifications may be established for the skew-symmetric problem of circular hole. Ignoring terms of order a^4/c^4 in σ_r, $\tau_{r\theta}$ and of order a^4/c^3 in u_r, v_θ, the supplementary terms to equations (13) and (14) can be obtained from

* Equations (38) may be inserted into the complex combinations, equations (64) in Appendix 1 to give equations (36) and (37).

equations (25) and (26) as

$$\Delta\sigma_r = \tau\left\{[4\sin 2\theta]\left(\frac{a}{c}\right)^2 + O\left[\left(\frac{a}{c}\right)^4\right]\right\}, \tag{40a}$$

$$\Delta\tau_{r\theta} = \tau\left\{[-2\cos 2\theta]\left(\frac{a}{c}\right)^2 + O\left[\left(\frac{a}{c}\right)^4\right]\right\}, \tag{40b}$$

and

$$\Delta u_r = \frac{\tau}{2\mu}c\left\{[(\kappa+1)\sin 2\theta]\left(\frac{a}{c}\right)^2 + O\left[\left(\frac{a}{c}\right)^4\right]\right\}, \tag{41a}$$

$$\Delta v_\theta = \frac{\tau}{2\mu}c\left\{-[(\kappa-1)\cos 2\theta]\left(\frac{a}{c}\right)^2 + O\left[\left(\frac{a}{c}\right)^4\right]\right\}. \tag{41b}$$

The extension of equations (40) and (41) to handle problems with cavities of any shape can be made by introducing the complex functions

$$\phi(z) = iBz^3, \qquad \psi(z) = 2iCz \tag{42}$$

from which the corrections on the stresses and displacements are calculated. They are given by

$$\Delta\sigma_r = 2C\sin 2\theta, \tag{43a}$$

$$\Delta\tau_{r\theta} = (6Br^2+2C)\cos 2\theta, \tag{43b}$$

and

$$\Delta u_r = \frac{1}{2\mu}r[(3-\kappa)Br^2+2C]\sin 2\theta, \tag{44a}$$

$$\Delta v_\theta = \frac{1}{2\mu}r[(\kappa+3)Br^2+2C]\cos 2\theta. \tag{44b}$$

For a round hole, the constants B and C take the values

$$B = -\frac{\tau}{c^2}\left(\frac{a}{c}\right)^2, \qquad C = 2\tau\left(\frac{a}{c}\right)^2.$$

It is appropriate, at this point, to propose a method for evaluating the stored energy in an infinite body, which may be summarized as follows:

> "The strain energy in an infinite elastic medium, loaded at infinity and weakened by cavities of certain shape, may be computed from the integral
>
> $$W = \tfrac{1}{2}\int_0^{2\pi} [(\sigma_r^{(1)}+\Delta\sigma_r)(u_r^{(1)}+\Delta u_r)+\tau_{r\theta}^{(1)}+\Delta\tau_{r\theta})(v_\theta^{(1)}+\Delta v_\theta)]_{r=c}\, c\,\mathrm{d}\theta \tag{45}$$
>
> where $\sigma_r^{(1)}$, $\tau_{r\theta}^{(1)}$ and $u_r^{(1)}$, $v_\theta^{(1)}$ are the respective stresses and displacements on a circle of radius r about the cavity, and the radius, c, of the closed contour can be made arbitrarily large. The corrections $\Delta\sigma_r$, $\Delta\tau_{r\theta}$ and Δu_r, Δv_θ for biaxial tension and/or pure shear applied at infinity correspond to those shown in equations (36), (37), (43) and (44)."

If the boundary, on which tractions are specified, has finite dimensions, the calculation of strain energy is straightforward. Complication arises only when the loaded boundary is of infinite extent. Surprisingly enough, this kind of problem has yet to be treated satisfactorily in the literature. To illustrate the present method of solution, equation (45) will be applied to problems with non-circular cavities.

ELLIPTICAL CAVITY

The problem of an elliptical cavity embedded in an infinite medium, Fig. 3, is of considerable importance in the development of fracture theories. The solution to this problem

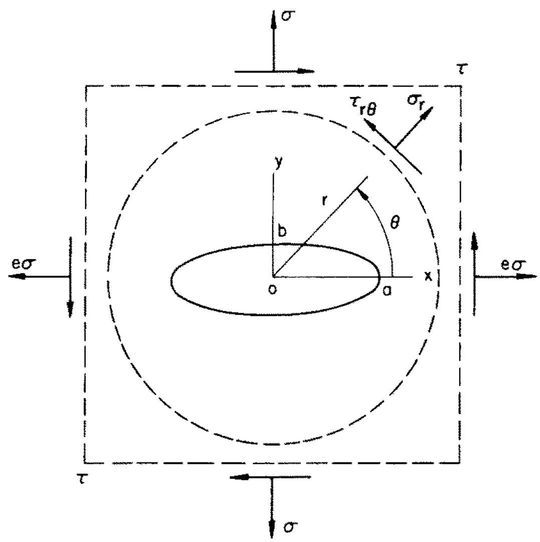

FIG. 3. Infinite plate with an elliptical cavity.

can be found in the work of Muskhelishvili [10]. He considers a conformal transformation from the z-plane, in which the cavity is elliptical in shape, onto the ζ-plane, in which the cavity becomes circular, by means of the mapping function

$$z = \omega(\zeta) = n\left(\zeta+\frac{m}{\zeta}\right), \qquad n > 0, \qquad 0 \leq m \leq 1.$$

The parameters m and n are related to the semi-axes of the ellipse by the relations

$$m = \frac{a-b}{a+b}, \qquad n = \frac{a+b}{2}. \tag{46}$$

Because the subsequent energy calculation requires only the knowledge of σ_r, $\tau_{r\theta}$ and u_r, v_θ in a set of cylindrical polar coordinate system, it suffices to cite

$$\sigma_r - i\tau_{r\theta} = 2\mathrm{Re}\left[\frac{\phi'(\zeta)}{\omega'(\zeta)}\right] - \frac{e^{2i\theta}}{[\omega'(\zeta)]^3}\{\bar{z}[\omega'(\zeta)\phi''(\zeta)-\phi'(\zeta)\omega''(\zeta)]+\psi'(\zeta)[\omega'(\zeta)]^2\}, \tag{47}$$

and

$$2\mu(u_r+iv_\theta) = e^{-i\theta}\left[\kappa\phi(\zeta)-\frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\phi'(\zeta)}-\overline{\psi(\zeta)}\right], \tag{48}$$

in which the two unknown functions $\phi(\zeta)$ and $\psi(\zeta)$ are determined from the boundary conditions stated in equations (19). The symmetrical and skew-symmetrical loadings at infinity will be treated individually.

Symmetrical loading

Let the edge of the elliptical opening be free from tractions and let the state of stress at infinity be tension of magnitude $e\sigma$ in the x-direction and σ in the y-direction. For this problem, the stress functions are [10]

$$\phi(\zeta) = \frac{\sigma n}{4}\left[(1+e)\left(\zeta-\frac{m}{\zeta}\right)-\frac{2(1-e)}{\zeta}\right], \tag{49a}$$

$$\psi(\zeta) = \frac{\sigma n}{2m}\left\{(1-e)\left(\frac{m\zeta^2+1}{\zeta}\right)-(1+m^2)[m(1+e)+(1-e)]\frac{\zeta}{\zeta^2-m}\right\}. \tag{49b}$$

The inverse of the mapping function is

$$\zeta = \frac{z}{2n}+\sqrt{\left[\left(\frac{z}{2n}\right)^2-m\right]}. \tag{50}$$

Making use of equations (47), (49), (50) and putting $z = re^{i\theta}$, the radial and shear stresses, expressed in series forms, are

$$\sigma_r^{(1)} = \frac{\sigma}{2}\Big\langle(1+e)-(1-e)\cos 2\theta-\{[(1+m^2)(1+e)+2m(1-e)]-4[m(1+e)+(1-e)\cos 2\theta\}\times\left(\frac{n}{r}\right)^2+O\left[\left(\frac{n}{r}\right)^4\right]\Big\rangle \tag{51a}$$

$$\tau_{r\theta}^{(1)} = \frac{\sigma}{2}\left\{(1-e)\sin 2\theta+2[m(1+e)+(1-e)]\left(\frac{n}{r}\right)^2+O\left[\left(\frac{n}{r}\right)^4\right]\right\}. \tag{51b}$$

Similarly, equations (48) and (49) yield the series expansions of the displacements in powers of the ratio n/r:

$$u_r^{(1)} = \frac{\sigma}{2\mu}r\Big\langle(1+e)(\kappa-1)-2(1-e)\cos 2\theta+2\{(1+m^2)(1+e)+2m(1-e)-(\kappa+1)[m(1+e)+(1-e)\cos 2\theta]\}\left(\frac{n}{r}\right)^2+O\left[\left(\frac{n}{r}\right)^4\right]\Big\rangle, \tag{52a}$$

$$v_\theta^{(1)} = \frac{\sigma}{2\mu}r\Big\langle 2(1-e)\sin 2\theta+\{(\kappa-1)[m(1+e)+(1-e)]\sin 2\theta\}\left(\frac{n}{r}\right)^2+O\left[\left(\frac{n}{r}\right)^4\right]\Big\rangle. \tag{52b}$$

An energy calculation, based on the stresses and displacements as they appear in equations (51) and (52), gives

$$W = W_0 + \Delta W$$

where W_0 corresponds to equation (10) and

$$\Delta W = \frac{\pi(1+\nu)\sigma^2 n^2}{8E}\{(3-\kappa)(1+e)[(1+m^2)(1+e)+2m(1-e)] \\ +2(\kappa-1)(1-e)[m(1+e)+(1-e)]\}, \quad (1921). \tag{53}$$

If $m = 0$ and $n = a$, equation (11) is recovered. In the special case of $m = 1$ and $n = a/2$, equation (53) reduces to the solution of a line crack, equation (12). A further simplification of having $e = 1$ renders equation (8) in [1] as obtained by Griffith in 1921. The mere fact that ΔW fails to check with the correct solution, equation (24), for a circular hole indicates that equation (53) is incorrect.

The modifications that must be imposed on equations (51) and (52) are given by equations (36) and (37), respectively. The obtained expressions are then adjusted through the constants A, B, and C such that the stresses satisfy equations (19) with $\tau = 0$. This requires

$$A = \frac{\sigma}{4}[(1+m^2)(1+e)+2m(1-e)]\left(\frac{n}{c}\right)^2, \qquad B = -\frac{\sigma}{2c^2}[m(1+e)+(1-e)]\left(\frac{n}{c}\right)^2,$$
$$C = \sigma[m(1+e)+(1-e)]\left(\frac{n}{c}\right)^2.$$

Hence, the integrand in equation (45) is completely known and the integration may be carried out to obtain that portion of the energy due to the elliptical cavity:

$$\Delta W = \frac{\pi(1+\nu)\sigma^2 n^2}{8E}(\kappa+1)[(1+m^2)(1+e)^2+4m(1-e^2)+2(1-e)^2]. \quad (1924)$$

By suitable choice of m and n, ellipse of any dimension and shape may be realized. However, it is more convenient to express ΔW in terms of a and b, the semi-axes of the ellipse. The parameters m and n can be eliminated by using equation (46) to find

$$\Delta W = \frac{\pi(1+\nu)\sigma^2}{16E}(\kappa+1)[(1+e)^2(a^2+b^2)+2(1-e^2)(a^2-b^2)+(1-e)^2(a+b)^2], \quad (1924) \tag{54}$$

which is in agreement with equation (24) for $a = b$. When $b = 0$, equation (54) takes the form of equation (3b) and is independent of e. This implies that ΔW is positive definite and is not affected by the applied stress in line with the crack edges. The energy of Griffith's crack corresponds to $e = 1$, i.e. a state of uniform stress at infinity.

Skew-symmetrical loading

The stress functions [10]

$$\phi(\zeta) = \frac{i\tau n}{\zeta}, \qquad \psi(\zeta) = \frac{i\tau n}{m}\left[\frac{m\zeta^2-1}{\zeta}+(1+m^2)\frac{\zeta}{\zeta^2-m}\right], \tag{55}$$

correspond to a state of pure shear parallel to the axes of the elliptical hole. To find the stresses and displacements in the medium, equations (47), (48), (50) and (55) may be combined to give

$$\sigma_r^{(1)} = \tau\left\{\sin 2\theta - [4\sin 2\theta]\left(\frac{n}{r}\right)^2 + O\left[\left(\frac{n}{r}\right)^4\right]\right\}, \tag{56a}$$

$$\tau_{r\theta}^{(1)} = \tau\left\{\cos 2\theta + [2\cos 2\theta]\left(\frac{n}{r}\right)^2 + O\left[\left(\frac{n}{r}\right)^4\right]\right\}, \tag{56b}$$

and

$$u_r^{(1)} = \frac{2\tau}{\mu}r\left\{\sin 2\theta + [(\kappa+1)\sin 2\theta]\left(\frac{n}{r}\right)^2 + O\left[\left(\frac{n}{r}\right)^4\right]\right\}, \tag{57a}$$

$$v_\theta^{(1)} = \frac{2\tau}{\mu}r\left\{\cos 2\theta + [(\kappa-1)\cos 2\theta]\left(\frac{n}{r}\right)^2 + O\left[\left(\frac{n}{r}\right)^4\right]\right\}. \tag{57b}$$

If modifications are not imposed on equations (56) and (57), the difference between the total energy of the system and the strain energy of the body with no hole would be

$$\Delta W = \frac{\pi(1+\nu)\tau^2 n^2}{E}(\kappa-1), \qquad (1921) \tag{58}$$

which is incorrect since it disagrees with equation (82) for $n = a$, the exact solution to the circular hole problem.

The present method of analysis requires the superposition of equations (43) and (44) onto equations (56) and (57), respectively. The constants B, C are evaluated from the conditions prescribed in equations (19) with $\sigma = 0$, and the results are

$$B = -\frac{\tau}{c^2}\left(\frac{n}{c}\right)^2, \qquad C = 2\tau\left(\frac{n}{c}\right)^2.$$

It follows from equation (45) that the integration around the circular contour of radius c yields

$$W = \frac{\pi(1+\nu)\tau^2 c^2}{E}\left\{1 + (\kappa+1)\left(\frac{n}{c}\right)^2 + O\left[\left(\frac{n}{c}\right)^4\right]\right\},$$

where $n = (a+b)/2$. The non-vanishing term that remains finite as $c \to \infty$ is

$$\Delta W = \frac{\pi(1+\nu)\tau^2}{4E}(\kappa+1)(a+b)^2, \qquad (1924). \tag{59}$$

It is surmised that equations (54) and (59) could also be obtained as limiting cases of the problem of confocal ellipses. Moreover, the formulation of energy criterion should be based upon equations (54), (59) and not equations (53), (58).

ENERGY BALANCE

The classical treatment of the fracture problem as originated by Griffith [1, 3] assumes the model of a line crack in an infinite medium. This model will be extended to the case of an elliptically-shaped defect. Griffith's energy approach considers the stored energy and surface energy in the system. The former arises from the work done by external loads, while the latter arises from the energy required for the formation of fracture surfaces as the original defect increases in size. Symbolically, Griffith's criterion of fracture may be written as

$$\frac{\partial}{\partial a}(\Delta S - \Delta W) = 0. \tag{60}$$

In equation (60), ΔS is equal to the product of the surface area of the ellipse and the specific surface energy of the material γ, i.e.

$$\Delta S = 4\gamma \int_0^{2\pi} \sqrt{[a^2 \cos^2\theta + b^2 \sin^2\theta]}\, d\theta = 4\gamma a E(k). \tag{61}$$

Here, $E(k)$ is the complete elliptic integral of the second kind with modulus

$$k = \sqrt{\left[1 - \left(\frac{b}{a}\right)^2\right]}$$

and its derivative with respect to a is

$$\frac{\partial E(k)}{\partial a} = \frac{1}{2a}\left(\frac{k'}{k}\right)^2 [E(k) - K(k)],$$

where $K(k)$ is the complete elliptic integral of the first kind associated with the argument k, which is related to k' by

$$k^2 + k'^2 = 1.$$

It will be assumed that the radius of curvature, $\rho = b^2/a$, at the ends of the major axis of the ellipse remains unchanged as the length of the axis is increased, i.e. $\partial\rho/\partial a = 0$. The excess of strain energy ΔW due to the elliptical cavity under biaxial tension and pure shear are given by equations (54) and (59), respectively.

Biaxial tension

Substituting equations (54) and (61) into the fracture criterion equation (60), the critical stress for failure is found:

$$\frac{\pi a(1+\nu)(\kappa+1)}{8\gamma E}\sigma_{cr}^2 = \frac{4}{k^2}\left[\frac{(1+k^2)E(k) - k'^2 K(k)}{(1+e)^2(3-k^2) + 2(1-e^2)(1+k^2) + (1-e)^2(1+k')(2+k')}\right], \tag{62}$$

which includes equation (2) for a sharp crack as a special case. The variations of the left hand side of equation (62) with the ratio b/a for $e = 1, 2$ are shown by the curves in Fig. 4. For $e = 1$, the critical stress for a narrow ellipse and a sharp crack changes only slightly. On the other hand, as the tension in the x-direction is increased, σ_{cr} tends to drop appreciably for a small variation of b/a about the origin. This effect is somewhat illustrated by the curve for $e = 2$.

Pure shear

Applying the failure criterion, equation (60), to the problem of uniform shear, equations (59) and (61) lead to the relation

$$\frac{\pi a(1+\nu)(\kappa+1)}{8\gamma E}\tau_{cr}^2 = \frac{(1+k^2)E(k) - k'^2 K(k)}{k^2(1+k')(2+k')}. \tag{63}$$

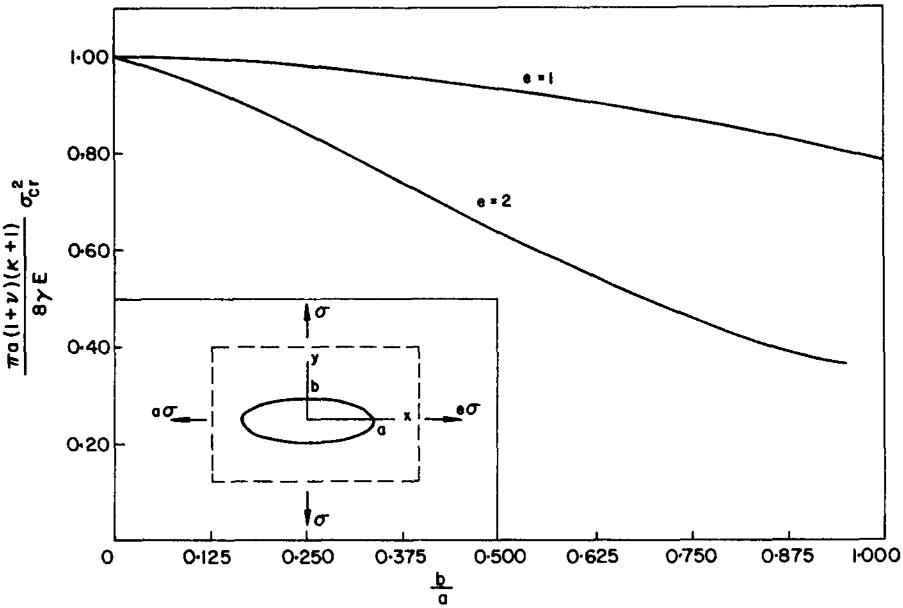

FIG. 4. Critical stress for biaxial tension.

Equation (63) shows that τ_{cr} is a complicated function of the geometry of the ellipse and its dependency on b/a can be best illustrated by the curve in Fig. 5. For small values of b/a,

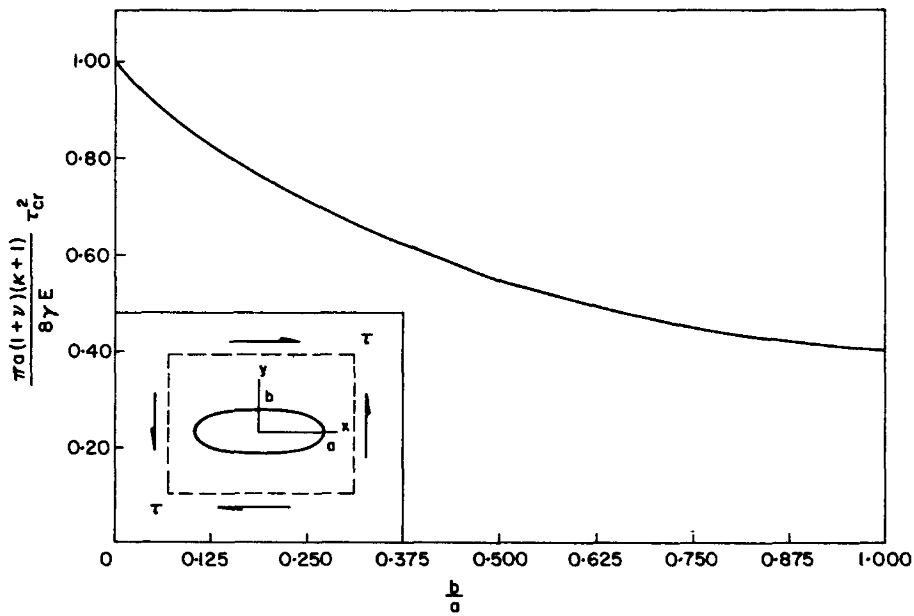

FIG. 5. Critical stress for pure shear.

the critical shear stress

$$\tau_{cr} = \sqrt{\left[\frac{8\gamma E}{\pi a(1+\nu)(\kappa+1)}\right]}, \qquad (1924)$$

of a line crack differs significantly from that of a narrow ellipse.

CONCLUSIONS

A method for the analytical determination of strain energy in an infinite body with a void of some kind is presented. Energy expressions are derived for an elliptical cavity subjected to normal and shear loads at infinity and are incorporated into a failure criterion. It appears that the present method of analysis is considerably simpler than the use of curvilinear coordinates as it was done in the classical works of Griffith [1, 3]. Consequently, it is also possible to calculate the energy of cavities of more general shapes such as the case of an ovaloid hole.

The most important result may not necessarily be the method of finding strain energy in an infinite medium, but the primary contribution here is perhaps that the disagreement between Griffith's results published in 1921 and 1924 is resolved and the way is cleared for the correct application of Griffith's concepts to fracture theories.

REFERENCES

[1] A. A. GRIFFITH, The phenomena of rupture and flow in solids. *Phil. Trans. R. Soc.* **A221**, 163 (1921).
[2] C. E. INGLIS, Stresses in a plate due to the presence of cracks and sharp corners. *Trans. Instn nav. Archit.* **60**, 219 (1913).
[3] A. A. GRIFFITH, The theory of rupture. *Proc. 1st Int. Congr. Appl. Mech.*, Delft, 1924, pp. 55–63.
[4] K. WOLF, Zur Bruchtheorie von A. Griffith. *Z. angew. Math. Mech.* **3**, 107 (1922).
[5] A. SMEKAL, Engineering strength and molecular strength. *Naturwissenschaften* **10**, 799 (1922).
[6] J. P. BERRY, General theory of brittle fracture. *Fracture Processes in Polymeric Solids*, edited by B. ROSEN, pp. 157–193. Interscience (1964).
[7] J. L. SMEDLOW, On Griffith's theory of fracture. *Int. J. fract. Mech.* **1**, 210 (1965).
[8] A. J. M. SPENCER, On the energy of the Griffith crack. *Int. J. engng Sci.* **3**, 441 (1965).
[9] I. S. SOKOLNIKOFF, *Mathematical Theory of Elasticity*, p. 86. McGraw-Hill (1956).
[10] N. I. MUSKHELISHVILI, *Some Basic Problems of the Mathematical Theory of Elasticity*, pp. 333–338. Noordhoff (1953).

APPENDIX 1

Complex Variable Formulation of the Annulus Problem

The term "annulus" will refer to the region between two concentric circumferences. The external and internal radii of the annulus are denoted by c and a, respectively, as shown in Fig. 2. This problem will be formulated in the complex plane $z = x+iy$, where the stresses* and displacements may be expressed in terms of two complex potentials $\phi(z)$, $\psi(z)$:

* The relation

$$\sigma_r+\sigma_\theta = 2[\phi'(z)+\overline{\phi'(z)}]$$

is not needed in the present analysis.

$$\sigma_r - i\tau_{r\theta} = \phi'(z) + \overline{\phi'(z)} - e^{2i\theta}[\bar{z}\phi''(z) + \psi'(z)], \tag{64a}$$

$$2(u_r + iv_\theta) = e^{-i\theta}[\kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}]. \tag{64b}$$

It is expedient to expand $\phi'(z)$ and $\psi'(z)$ in Laurent series as follows:

$$\phi'(z) = \sum_{-\infty}^{\infty} A_p z^p, \qquad \psi'(z) = \sum_{-\infty}^{\infty} B_p z^p. \tag{65}$$

These functions are holomorphic in the annulus, where the inner circumference may be shrunk into a single point and the outer circumference may become infinitely large. The logarithmic terms in equation (64b) must be eliminated to ensure single-valuedness of the displacements. Hence, it is necessary to have

$$\kappa A_{-1} + \bar{B}_{-1} = 0. \tag{66}$$

Similar relations for the remaining coefficients A_p, B_p may be established from the boundary conditions of the problem.

Suppose that the annulus is internally free from applied loads, i.e.

$$(\sigma_r - i\tau_{r\theta})_{r=a} = 0 \tag{67a}$$

and is externally subjected to surface tractions, which may be represented by the complex Fourier series

$$(\sigma_r - i\tau_{r\theta})_{r=c} = \sum_{-\infty}^{\infty} C_p e^{ip\theta}. \tag{67b}$$

The coefficients C_p are determined by the formula

$$C_p = \frac{1}{2\pi}\int_0^{2\pi} [\sigma_r(\theta) - i\tau_{r\theta}(\theta)]_{r=c}\, e^{-ip\theta}\, d\theta, \qquad p = 0, \pm 1, \pm 2, \ldots. \tag{68}$$

Combining equations (64a) and (67) gives

$$\sum_{-\infty}^{\infty} \left\{(1-p)A_p - B_{p-2}\begin{bmatrix} a^{-2} \\ c^{-2}\end{bmatrix}\right\}\begin{bmatrix} a^p \\ c^p\end{bmatrix} e^{ip\theta} + \sum_{-\infty}^{\infty} \bar{A}_p \begin{bmatrix} a^p \\ c^p\end{bmatrix} e^{-ip\theta} = \begin{bmatrix} 0 \\ C_p e^{ip\theta}\end{bmatrix}.$$

The coefficients of like powers of $e^{i\theta}$ on both sides of the above expressions are equated to yield for $p = 0$,

$$(A_0 + \bar{A}_0) - B_{-2}\begin{bmatrix} a^{-2} \\ c^{-2}\end{bmatrix} = \begin{bmatrix} 0 \\ C_0\end{bmatrix}, \tag{69}$$

and for $p = \pm 1, \pm 2, \ldots$,

$$(1-p)A_p\begin{bmatrix} a^p \\ c^p\end{bmatrix} - B_{p-2}\begin{bmatrix} a^{p-2} \\ c^{p-2}\end{bmatrix} + \bar{A}_{-p}\begin{bmatrix} a^{-p} \\ c^{-p}\end{bmatrix} = \begin{bmatrix} 0 \\ C_p\end{bmatrix}. \tag{70}$$

Since the addition of an imaginary constant of $\phi'(z)$ cannot affect the state of stress, A_0 may be taken as real, i.e. $A_0 = \bar{A}_0$. Now, the constants A_0 and B_{-2} in equations (69) may be solved:

$$A_0 = \frac{1}{2}\left(\frac{c^2}{c^2-a^2}\right)C_0, \qquad B_{-2} = \left(\frac{a^2c^2}{c^2-a^2}\right)C_0. \tag{71}$$

Moreover, the first of equations (70) demands that for $p = 1$,

$$\bar{A}_{-1} - B_{-1} = 0, \quad \text{or} \quad A_{-1} = \bar{B}_{-1}$$

which contradicts equation (66) unless both A_{-1} and B_{-1} are identically zero. If the tractions on the boundary $r = c$ are self-equilibrating, C_1 must also vanish. Solving equations (70), it is found that

$$A_1 = \left(\frac{c^3}{c^4-a^4}\right)\bar{C}_{-1} \tag{72}$$

and

$$A_p = \{(1+p)(c^2-a^2)c^{-p}C_p - [c^{2(1-p)} - a^{2(1-p)}]c^p\bar{C}_{-p}\}(c^2/D_p), \qquad p = \pm 2, \pm 3, \ldots, \tag{73}$$

where

$$D_p = (1-p^2)(c^2-a^2)^2 - [c^{2(1+p)} - a^{2(1+p)}][c^{2(1-p)} - a^{2(1-p)}].$$

In the same fashion, B_p are obtained:

$$B_{p-2} = (1-p)a^2A_p + a^{2(1-p)}\bar{A}_{-p}, \qquad p = \pm 2, \pm 3, \ldots. \tag{74}$$

Once the constants C_p are determined from equation (68), the coefficients A_p, B_p follow immediately from equations (71) to (74). Thus, the problem is basically solved.

APPENDIX 2

Stresses Even in $\sigma_r(-\theta) = \sigma_r(\theta)$; Odd in $\tau_{r\theta}(-\theta) = -\tau_{r\theta}(\theta)$

Let the outer circumference of the annulus be subjected to the stresses

$$(\sigma_r - i\tau_{r\theta})_{r=c} = \frac{\sigma}{2}(1+e) - \frac{\sigma}{2}(1-e)e^{2i\theta}$$

Then, equation (68) leads to

$$C_0 = \frac{\sigma}{2}(1+e), \qquad C_2 = -\frac{\sigma}{2}(1-e), \quad \text{and} \quad C_p = 0, \quad \text{for } p = \pm 1, -2, \pm 3, \ldots.$$

From equations (71) to (74), the non-vanishing coefficients of A_p, B_p are

$$A_{-2} = \frac{\sigma(1-e)a^2c^2(a^4+a^2c^2+c^4)}{2(c^2-a^2)^3}, \qquad A_0 = \frac{\sigma(1+e)c^2}{2(c^2-a^2)}, \qquad A_2 = -\frac{3\sigma(1-e)a^2c^2}{2(c^2-a^2)^3},$$

and

$$B_{-4} = \frac{3\sigma(1-e)a^4c^4(a^2+c^2)}{2(c^2-a^2)^3}, \qquad B_{-2} = \frac{\sigma(1+e)a^2c^2}{2(c^2-a^2)}, \qquad B_0 = \frac{\sigma(1-e)c^2(4a^4+a^2c^2+c^4)}{2(c^2-a^2)^3},$$

which may be inserted into equations (65) to render the Goursat functions

$$\phi(z) = \frac{\sigma a^2c^2}{4(c^2-a^2)^3}\left[(1+e)\left(\frac{c^2}{a^2}-1\right)^2 z-2(1-e)z^3-2(1-e)(a^4+a^2c^2+c^4)\frac{1}{z}\right], \tag{75a}$$

$$\psi(z) = \frac{\sigma a^2c^2}{2(c^2-a^2)^3}\left[(1-e)\left(4a^2+c^2+\frac{c^4}{a^2}\right)z-(1+e)(c^2-a^2)^2\frac{1}{z} -(1+e)a^2c^2(a^2+c^2)\frac{1}{z^3}\right]. \tag{75b}$$

By letting the radius c to become infinite, equations (75) reduce to

$$\phi(z) = \frac{\sigma}{2}\left[\frac{1}{2}(1+e)z-\frac{(1-e)a^2}{z}\right], \tag{76a}$$

$$\psi(z) = \frac{\sigma}{2}\left[(1-e)z-\frac{(1+e)a^2}{z}-\frac{(1-e)a^4}{z^3}\right]. \tag{76b}$$

Equations (76) represent the solution to the problem of a circular hole in an infinite medium under biaxial tension.

APPENDIX 3

Stresses Odd in $\sigma_r(-\theta) = -\sigma_r(\theta)$; Even in $\tau_{r\theta}(-\theta) = \tau_{r\theta}(\theta)$

When the stress combination

$$(\sigma_r - i\tau_{r\theta})_{r=c} = -i\tau e^{2i\theta}$$

is substituted into equation (68), all coefficients in the Fourier expansion, equation (67b), vanish except

$$C_2 = -i\tau.$$

The corresponding coefficients in the Laurent series, equations (65), are

$$A_{-2} = -\frac{i\tau a^2c^2(a^4+a^2c^2+c^4)}{(c^2-a^2)^3}, \qquad A_2 = -\frac{3i\tau a^2c^2}{(c^2-a^2)^3},$$

and

$$B_{-4} = -\frac{3i\tau a^4c^4(a^2+c^2)}{(c^2-a^2)^3}, \qquad B_0 = \frac{i\tau c^2(4a^4+a^2c^2+c^4)}{(c^2-a^2)^3}.$$

Hence,

$$\phi(z) = -\frac{i\tau a^2c^2}{(c^2-a^2)^3}\left[z^3-(a^4+a^2c^2+c^4)\frac{1}{z}\right], \tag{77a}$$

$$\psi(z) = \frac{i\tau a^2c^2}{(c^2-a^2)^3}\left[\left(4a^2+c^2+\frac{c^4}{a^2}\right)z+a^2c^2(a^2+c^2)\frac{1}{z^3}\right], \tag{77b}$$

The Goursat functions for the case of an infinite plate with a round hole subjected to pure shear are obtained from equations (77) by setting $c = \infty$ and are given by

$$\phi(z) = \frac{i\tau a^2}{z}, \qquad \psi(z) = i\tau\left(z + \frac{a^4}{z^3}\right). \tag{78}$$

(*Received* 11 *August* 1966)

Résumé—La théorie de Griffith relative à la propagation d'une fissure instable est de nouveau examinée pour étudier les différences d'opinion concernant la dépendance précise de l'effort critique sur la constante élastique dans une matière fragile. Le désaccord résultant du calcul de l'énergie de tension contenue dans un corps fêlé chargé à l'infinité est résolu en observant que l'énergie de la fissure ou cavité d'une certaine forme est géométriquement indépendante. L'on a déduit de la solution du problème de l'anneau concentrique que les efforts et déplacements sur un contour fermé aux environs de la cavité doivent être modifiés pour produire la forme correct d'énergie élastique. La théorie de Clapeyron est appliquée de façon que l'on puisse dérivér la fonction d'énergie à partir du travail effectué par les tractions de surface. Une méthode générale pour la détermination de l'énergie de tension dans un milieu infini, ayant des cavités d'une configuration arbitraire est présentée. Des solutions de formes fermées sont obtenues pour la résolution du problème d'une fissure de forme elliptique et sont incorporées dans une théorie de fracture de corps fragile. Plus précisément, des charges critiques pour une fissure elliptique causée par une tension biaxiale et cisaillement pur y sont donnés. Cette analyse actuelle confirme également les revendications de Griffith en 1924 que son expression de l'énergie originale, publiée en 1921 est en effet erronée.

Zusammenfassung—Die Theorie von Griffith für das labile Vordringen des Rißes wird untersucht um die verschiedenen Ansichten über die genauen Zusammenhänge zwischen der kritischen Spannung und der elastischen Konstante in sprödem Meterial festzustellen. Die anscheinende Meinungsverschiedenheit die aus der Berechnung der Formänderungsenergie eines gerißenen Körpers bei Unendlichkeit, die aufgespeichert ist entsteht wird durch die Beobachtung gelöst, daß die Riß- oder Hohlkörpernergie gewisser Formen geometrisch unabhängig ist. Aus der Lösung des Konzentrischen-Kreisringproblemes wird festgestellt, daß die Spannungen und Verschiebungen einer geschlossenen Kurve um den Hohlkörper geändert werden müssen um die korrekte elastische Energieform zu ergeben. Das Theorem Clapeyrons wird angewandt um die Kraftfunktion aus der geleisteten Arbeit abzuleiten. Eine allgemeine Methode zur Feststellung der Spannungsenergie in einem unendlichen Medium mit Hohlkörpern beliebiger Form wird gegeben. Geschlossene Lösungen werden für das Problem der elliptischen Risse gegeben und in die Theorie der Sprödbrüche eingereiht. Insbesondere werden die kritischen Spannungen für eine elliptische Riß- oder Bruchstelle als Folge biaxialer Spannung und reiner Scherung gegeben. Die gegebene Analyse bestätigt auch Griffith's Ansicht, (1924) daß sein ursprünglicher Energiesatz, wie in 1921 veröffentlicht ferhlerhaft war.

Абстракт—Пересматривается теория Гриффитса для распространения неустойчивой трещины, чтобы исследовать разницу мнений относительно точной зависимости критического напряжения и эластической постоянной в хрупком материале. Видимое несогласие, возникающее из вычисления энергии деформации, накопляющейся в треснутом теле, нагружённом в бесконечности, разрешается наблюдением того, что энергия трещины или полости определённой формы умеренно геометрически независима. Из решения проблемы концентрического кругового кольца найдено, что напряжения и смещения на замкнутом контуре около полости должны быть изменены, чтобы уступить правильной форме эластической энергии. Теорема Клапейрона употребляется так, что функция энергии может быть выведена из работы сил сцепления поверхности. Предлагается общий метод определения энергии деформации в бесконечной среде с полостями произвольной конфигурации. Получены решения замкнутой формы проблемы эллиптически-образного дефекта и объединены в теорию хрупкого излома. Более специфично даются критические напряжения для эллиптического дефекта трещины по причине биаксиального натяжения и чистого сдвига. Настоящий анализ подтверждает также утверждение Гриффитса в 1924-м году, что его первоначальное выражение энергии, опубликованное в 1921-м году—действительно ошибочно.